VARIATIONAL PROBLEM OF HEATING THIN BODIES

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UDC 536.244

The problem of heating thin bodies optimally, in terms of a composite minimum fuel cost and metal loss by oxidation, is solved analytically. The solution is based on L. S. Pontryagin's maximum principle.

We consider the variational problem of heating thin bodies at a composite minimum fuel cost and metal loss by oxidation. The heat is transmitted from the flue gases to the metal according to Newton's Law and the heating dynamics is characterized by the following equation:

$$\frac{dT}{d\tau}=\frac{T_G-T}{\mu}, \tag{1}$$

where $u = Gc/F\alpha$.

The total quantity of heat B [1] and the thickness squared of oxide film ω^2 [2] after time τ_f are defined in terms of functionals

$$B=\int\limits_0^{\tau f}\left(M_0\frac{T_G-T}{T_c-T_G}+M_x\right)d\tau, \tag{2}$$

$$\omega^2=\int\limits_0^{\tau f}\frac{\varkappa}{T\exp(\beta/T)}\,d\tau, \tag{3}$$

where $M_0 = \alpha FT_c$.

It is required to design a control $T_G(\tau)$ which would, during the transition of a metal from the initial state T_i to the final state T_f, ensure a minimum of the optimality criterion R:

$$R=\min_{T_G}[KB(\tau_f)+\omega^2(\tau_f)], \tag{4}$$

where $K = k_T/k_0$.

In order to solve the problem, we apply the maximum principle. Using the mean value of specific heat c, of the heat transfer coefficient α, and of the idle-run power M_x, we can write the Hamiltonian for the problem:

$$H=P_1M_0\frac{T_G-T}{T_c-T_G}+P_1M_x+P_2\varkappa T^{-1}\exp\left(-\frac{\beta}{T}\right)+P_3\frac{T_G-T}{\mu}. \tag{5}$$

The conjugate variables P_1, P_2, and P_3 are determined from the following relations:

$$\frac{dP_1}{d\tau}=-\frac{\partial H}{\partial B};\quad \frac{dP_2}{d\tau}=-\frac{\partial H}{\partial(\omega^2)};\quad \frac{dP_3}{d\tau}=-\frac{\partial H}{\partial T}. \tag{6}$$

Solving Eq. (6) we have

$$P_1=-K; \tag{7}$$

$$P_2=-1; \tag{8}$$

Institute of Metallurgy, Dnepropetrovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 23, No. 3, pp. 545-549, September, 1972. Original article submitted November 16, 1971.

$$\frac{dP_3}{d\tau} = \frac{P_1 M_0}{T_c - T_G} - P_2 \varkappa \left(\frac{\beta}{T} - 1\right) T^{-2} \exp\left(-\frac{\beta}{T}\right) + \frac{P_3}{\mu}. \tag{9}$$

From the extremum condition $\partial H/\partial T_G = 0$ we find

$$P_3 = \mu K M_0 \frac{T_c - T}{(T_c - T_G)^2}. \tag{10}$$

We test the function H for a maximum:

$$\frac{\partial^2 H}{\partial T_G^2} = -2KM_0 \frac{T_c - T}{(T_c - T_G)^3}. \tag{11}$$

Here $K > 0$, $M_0 > 0$, and $T_c > T_G > T$ and, therefore, function H has a maximum.

We differentiate expression (10) with respect to time

$$\frac{dP_3}{d\tau} = \mu K M_0 \frac{2(T_c - T)\frac{dT_G}{d\tau} - (T_c - T_G)\frac{dT}{d\tau}}{(T_c - T_G)^3}. \tag{12}$$

Inserting the value of P_3 from (10) and of $dP_3/d\tau$ from (12) into Eq. (9), then solving (9) simultaneously with (1), we have

$$\frac{dT_G}{dT} = \frac{T_c - T_G}{T_c - T} + \frac{\varkappa}{2KM_0} \cdot \frac{(T_c - T_G)^3 \exp\left(-\frac{\beta}{T}\right)}{(T_c - T)(T_G - T)T^2} \left(\frac{\beta}{T} - 1\right). \tag{13}$$

We now change the variables:

$$y = T_c - T_G;\; z = T_c - T. \tag{14}$$

Taking into account (14), we can represent Eq. (13) in the following form:

$$\frac{dy}{dz} = \frac{y\left[(z-y)(T_c - z)^3 - A_1 (T_c - z) y^2 \exp\left(-\frac{\beta}{T_c - z}\right) + A_2 y^2 \exp\left(-\frac{\beta}{T_c - z}\right)\right]}{z(z-y)(T_c - z)^3}, \tag{15}$$

with

$$A_1 = \frac{\varkappa}{3KM_0}; \quad A_2 = \frac{\beta\varkappa}{2KM_0}.$$

We introduce the function

$$y = \frac{z^2 (T_c - z)^3}{x + z(T_c - z)^3}. \tag{16}$$

Differentiating Eq. (16) with respect to z and inserting the values of y and dy/dz into Eq. (15), we obtain the Bernoulli equation

$$\frac{dx}{dz} = \left[\frac{T_c - 4z}{z(T_c - z)}\right] x + \left\{z^2 (T_c - z)^3 [A_1 (T_c - z) - A_2] \exp\left(-\frac{\beta}{T_c - z}\right)\right\} \frac{1}{x}, \tag{17}$$

whose solution by well known methods is

$$x = z(T_c - z)^3 \sqrt{\frac{A_1}{T_c - z} \exp\left(-\frac{\beta}{T_c - z}\right) + C}. \tag{18}$$

Substituting this for x in Eq. (16) and changing to variables T_G, T will yield an expression which establishes the optimum relation between the temperature of flue gases and the temperature of metal during the heating process:

$$T_G = T_c - \frac{T_c - T}{1 + \sqrt{\frac{A_1}{T} \exp\left(-\frac{\beta}{T}\right) + C}}. \tag{19}$$

We will then use relation (19) for determining $T(\tau)$ from Eq. (1):

$$\tau = \mu \ln \frac{C_1 \cdot f(T)}{T_c - T}, \tag{20}$$

where

$$\ln f(T) = \int \frac{dT}{(T_c - T)\sqrt{\frac{A_1}{T}\exp\left(-\frac{\beta}{T}\right) + C}}. \tag{21}$$

The integration constants C and C_1 are evaluated from the initial conditions, and the heating curve for the metal is plotted according to Eq. (20). The optimum control $T_G(\tau)$ is determined from expression (19) with $T(\tau)$ already figuring in it.

The useful thermal power of the furnace M_f is determined according to the formula

$$M_f = M_0 \sqrt{\frac{A_1}{T}\exp\left(-\frac{\beta}{T}\right) + C}. \tag{22}$$

The heating time can be calculated from the condition that H = 0, unless it is limited by constraints of the problem. Expression (5) becomes in this case, with (7), (8), and (10) taken into account,

$$-KM_0 \frac{T_G - T}{T_c - T_G} - KM_x - \varkappa T^{-1} \exp\left(-\frac{\beta}{T}\right) + KM_0 \frac{(T_G - T)(T_c - T)}{(T_c - T_G)^2} = 0. \tag{23}$$

Having solved Eq. (23) for T_G, we obtain

$$T_G = \frac{T + T_c \sqrt{\frac{M_x}{M_0} + \frac{D}{T}\exp\left(-\frac{\beta}{T}\right)}}{1 + \sqrt{\frac{M_x}{M_0} + \frac{D}{T}\exp\left(-\frac{\beta}{T}\right)}}, \tag{24}$$

with $D = \varkappa / KM_0$.

After inserting (24) into Eq. (1) and integrating, we obtain a relation for the optimum heating period

$$\tau_f^* = \mu \ln \left[\frac{T_c - T_i}{T_c - T_f} \varphi(T_i, T_f)\right], \tag{25}$$

with

$$\ln \varphi(T_i, T_f) = \int_{T_i}^{T_f} \frac{dT}{(T_c - T)\sqrt{\frac{M_x}{M_0} + \frac{D}{T}\exp\left(-\frac{\beta}{T}\right)}}. \tag{26}$$

From these solutions follows the special case of optimally heating a metal at a minimum fuel consumption [1]. Thus, for example, if we let D = 0 in expression (26) ($k_0 = 0$, i.e., metal oxidation is negligible), we obtain a formula for the optimum heating period in terms of minimum fuel consumption:

$$\tau_f = \mu \left(1 + \sqrt{\frac{M_0}{M_x}}\right) \ln \frac{T_c - T_i}{T_c - T_f}. \tag{27}$$

If the control $T_G(\tau)$ is restricted to $T_G < T_0$, then the heating mode (when this restriction goes into effect) will be made up of two periods: 1) optimum heating in terms of a composite minimum fuel cost and metal loss by oxidation, and 2) heating at a constant flue-gas temperature. The metal temperature during the second period can be found from the expression

$$T = T_G - (T_G - T_{iI}) \exp\left(\frac{\tau_{iI} - \tau}{\mu}\right). \tag{28}$$

If one compares the optimum heating modes of thin bodies first in terms of minimum fuel consumption and then in terms of minimum loss of metal by oxidation, then in the first case the optimum heating period is of some length τ_f and in the second case the temperature rises at the maximum rate. In our problem the optimum heating period is shorter than in the first case and longer than in the second case, it also depends on the ratio of cost factors k_T and k_0.

NOTATION

T_G	is the temperature of flue gases;
T	is the temperature of heated body;
τ	is the time;
τ_f	is the completion time of heating process;
G	is the mass of heated body;
c	is the specific heat of heated material;
F	is the surface area of heated body;
α	is the heat transfer coefficient;
B	is the thermal flux;
ω^2	is the thickness squared of oxide film;
T_c	is the calculated temperature of fuel combustion [3];
M_x	is the idle-run furnace power;
T_i, T_f, T_{iI}	are the initial, final, and transition (between the two heating periods) temperature of the metal;
P_1, P_2, P_3	are the conjugate variables;
τ_{i1}	is the transition time (beginning of second heating period);
R	is the optimality criterion;
H	is the Hamilton function;
T_0	is the control limit;
k_T, k_0	are the cost factors of fuel consumption and scale formation respectively;
$\varkappa$, β	are the constant coefficients characterizing the oxidation kinetics.

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